Correspondence

Theory of Unidirectionality and Conjugate Matching in Three-Port Time-Varying Reactance Circuits

Abstract—**In recent years interest has been shown in parametric circuits having special properties such as unidirectionality and conjugate matching. This correspondence discusses the properties of a particular class of such circuits; viz., those which may be regarded as a three-port time-varying reactance circuit terminated by complex impedances. The theory is presented in terms of characteristic terminations, to which constraints imposed by the Manley-Rowe relations are applied. Applications of the theory include the synthesis of a three-port time-varying reactance circulator and of unidirectional and conjugately matched amplifiers.**

I. Introduction

In recent years a certain amount of interest has been shown in parametric circuits having special properties such as unidirectionality and conjugate matching with high gain [1]–[9]. Such circuits would eliminate the need for the ferrite circulators and isolators presently used in parametric amplifying devices. Many of these special circuits may be regarded as three-port time-varying reactance circuits terminated by resistive or complex impedances. A typical example might be a four-frequency parametric up-converter, where the three ports are the signal input port, the upper-sideband output port, and the lower-sideband or idler port, the termination of which controls the regeneration. Although the time-varying circuit may take various forms (e.g., combinations of pumped inductance and capacitance, two-diode circuits, or single-diode double-pumped circuits) a great deal can be deduced from the fact that the elements used obey the Manley-Rowe power relations. The purpose of this correspondence is to discuss the properties of general three-port time-varying reactance circuits, and to show how the frequencies of the ports may be chosen to fulfill special requirements.

II. Theory

A. The Manley-Rowe Constraint

We shall assume that the variation of the time-varying elements is periodic, but not necessarily sinusoidal, and that the frequencies of the ports are first-order sidebands of the pump frequency or its harmonics. Treating the frequencies of lower sidebands (that is, inverted sidebands) as negative, the angular frequency of each of the three ports can be written in the form $\omega_s + n\omega_p$, where n is a positive or negative integer and ω_p the fundamental pump angular frequency. Under these conditions it follows from the Manley-Rowe relations [10] that

$$\sum_{m=1}^{3} \frac{\text{power at } \omega_m}{\omega_m} = 0 \tag{1}$$

where the summation is taken over the three signal ports, ω_m being the angular frequency of port m. This relation holds not only for single time-varying reactances, but also for combinations of fixed and time-varying reactances.

An alternative way of expressing the Manley-Rowe relations is in terms of the admittance matrix of the network. If the network is represented by the admittance matrix

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{2}$$

it can be shown [11] that the Manley-Rowe relations correspond to the constraint

$$\frac{Y_{nm}}{j\omega_n} = \left[\frac{Y_{mn}}{j\omega_m}\right]^* \tag{3}$$

where Y_{nn} is purely imaginary. The asterisk signifies complex conjugate.

It is convenient to write the matrix as

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} j\omega_1 C_{11} & j\omega_1 C_{12} & j\omega_1 C_{13} \\ j\omega_2 C_{21} & j\omega_2 C_{22} & j\omega_2 C_{23} \\ j\omega_3 C_{31} & j\omega_3 C_{32} & j\omega_3 C_{33} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{4}$$

so that the constraint becomes

$$C_{nm} = C_{mn}^* \tag{5}$$

where C_{nn} is real. This method of writing should not be taken to mean that the network contains only capacitance; the C's will in general be frequency-dependent and will include the filters in the circuit.

B. Limitations of Two-Port Circuits

A two-port network may be regarded as a special case of the three-port network, and is represented by the admittance matrix equation

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{6}$$

In this connection, it should be noted that the term port is not necessarily synonymous with terminal-pair; if several frequency components appear at a single terminal-pair, each is treated as a separate port. Applying the restriction of (3) to (6) gives

$$\left|\frac{Y_{21}}{Y_{12}}\right| = \left|\frac{\omega_2}{\omega_1}\right|. \tag{7}$$

A two-port time-varying reactance circuit thus cannot be unidirectional.

C. Characteristic Terminations of a Three-Port Network

Before turning specifically to three-port time-varying circuits, we will consider how a general linear nonreciprocal three-port network can be used to produce a unidirectional device. The matrix equation of a three-port network can be written as in (2).

If the ports are terminated with admittances Y_1, Y_2, and Y_3, and current sources I_{10}, I_{20}, and I_{30} applied as in Fig. 1, the new matrix (Y') becomes

$$\begin{bmatrix} I_{10} \\ I_{20} \\ I_{30} \end{bmatrix} = \begin{bmatrix} Y_1 + Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_2 + Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_3 + Y_{33} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{8}$$

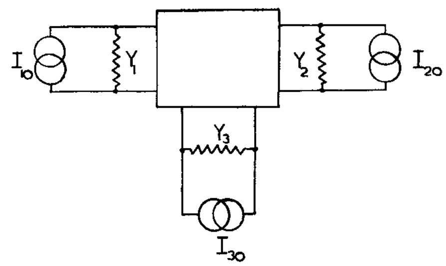

Fig. 1. Three-port network with terminations.

Inverting this gives

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} & \frac{A_{31}}{\Delta} \\ \frac{A_{12}}{\Delta} & \frac{A_{22}}{\Delta} & \frac{A_{32}}{\Delta} \\ \frac{A_{13}}{\Delta} & \frac{A_{23}}{\Delta} & \frac{A_{33}}{\Delta} \end{bmatrix} \begin{bmatrix} I_{10} \\ I_{20} \\ I_{30} \end{bmatrix} \tag{9}$$

where A_{mn} is the co-factor of Y_{mn}' in (8) and Δ is the determinant. Assuming $\Delta \neq 0$, the condition for zero gain from port 2 to port 1 is

$$A_{21} = 0. \tag{10}$$

This defines a value for Y_3 which is independent of Y_1 and Y_2 and is given by

$$Y_3 = -Y_{33} + \frac{Y_{13}Y_{32}}{Y_{12}}. \tag{11}$$

Alternatively, for zero gain from port 1 to port 2 we must put

$$Y_3 = -Y_{33} + \frac{Y_{23}Y_{31}}{Y_{21}}. \tag{12}$$

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Similar values can be defined for Y_1 and Y_2 by cyclic permutation of (11) and (12). Thus we can say that, with certain exceptions, each port of a three-port network has two characteristic terminating admittances which make the device unidirectional between the other two ports. To distinguish between them, we shall refer to them as cyclic or anticyclic, according to the direction of the *nonzero* gain.

D. Exceptions

a) Degenerate case: If the two characteristic terminations of a port coincide, we cannot make the reverse gain zero without at the same time making the forward gain zero. Under these conditions, the device cannot be made unidirectional. An example is a reciprocal three-port network, but degenerate three ports need not necessarily be reciprocal. The condition for degeneracy is, from (11) and (12),

$$Y_{12}Y_{23}Y_{31} = Y_{13}Y_{32}Y_{21}. \qquad (13)$$

This is symmetrical in 1, 2, and 3, and therefore if one port of the device is degenerate, all ports are degenerate.

b) Critical stability ($\Delta = 0$): Characteristic termination of a port may, depending on the terminations of the other two ports, cause the device to become critically stable instead of unidirectional. This is of particular importance when we wish to terminate more than one port characteristically. Suppose we terminate port 1 cyclically and port 2 anticyclically. Then we have

$$A_{32} = A_{31} = 0. \qquad (14)$$

From this it can be shown that A_{33} is also zero and hence

$$\Delta = 0. \qquad (15)$$

Thus termination of one port with its cyclic characteristic admittance and a second port with its anticyclic characteristic admittance produces critical stability.

E. The Characteristic Terminations of a Three-Port Time-Varying Reactance Circuit

The admittance matrix of a three-port time-varying reactance circuit may be written as in (4). Substituting from this matrix in (11) and (12) gives the characteristic terminating admittances of port 3

$$Y_3(\text{cyclic}) = -j\omega_3 C_{33} + \frac{j\omega_3 C_{13}C_{32}}{C_{12}} \qquad (16)$$

$$Y_3(\text{anticyclic}) = -j\omega_3 C_{33} + \frac{j\omega_3 C_{23}C_{31}}{C_{21}} \cdot \qquad (17)$$

Thus, since C_{mn} and C_{nm} are complex conjugates,

$$Y_3(\text{cyclic}) = -Y_3^*(\text{anticyclic}) \qquad (18)$$

and similarly for the other two ports. Thus the two characteristic terminations of a port have equal imaginary parts and equal and opposite real parts. Consequently only one can be passive. The condition for degeneracy is, from (13),

$$C_{12}C_{23}C_{31} = C_{13}C_{32}C_{21},$$

i.e.,

$$C_{12}C_{23}C_{31} = C_{31}^*C_{23}^*C_{12}^*. \qquad (19)$$

Hence $C_{12}C_{23}C_{31}$ is real or zero. An example of a degenerate circuit would be a multi-element circuit in which all the elements are pumped in phase, as it is then possible to choose the time origin such that all C_{mn} are real.

F. Active and Passive Characteristic Terminations

The phase angle of $C_{13}C_{32}/C_{12}$ in (16) is the same as that of $C_{13}C_{32}C_{21}$. Hence the real part of Y_3 (cyclic) has the same sign as $-\omega_3 \operatorname{Im} C_{13}C_{32}C_{21}$, which is the same as the sign of $\omega_3 \operatorname{Im} C_{12}C_{23}C_{31}$. This depends on the sign of ω_3, i.e., on whether ω_3 is a noninverted or an inverted sideband. Since $C_{12}C_{23}C_{31}$ is symmetrical in 1, 2, and 3, it is independent of the port we are considering. Thus if

$$\operatorname{Im} C_{12}C_{23}C_{31} > 0 \qquad (20)$$

the cyclic terminations of positive-frequency ports and the anticyclic terminations of negative-frequency ports are the passive ones. If

$$\operatorname{Im} C_{12}C_{23}C_{31} < 0 \qquad (21)$$

the reverse is true.

G. Conjugate Matching

A point of interest is that, if two ports are terminated either both with their cyclic or both with their anticyclic characteristic terminations, they will be conjugately matched, irrespective of the third termination. This can be seen as follows. We have already noted in Section II-D that if one port is terminated cyclically and a second port terminated anticyclically, critical stability results. Under this condition, the terminating admittance at each of the two ports is equal and opposite to the input admittance at that port. If we now replace the anticyclic termination by the corresponding cyclic termination, this merely involves changing the sign of the real part. The port is thus now conjugately matched. By a similar argument, the other port is also conjugately matched. Similarly, two anticyclic characteristic terminations are conjugately matched.

III. Application of the Theory to the Design of Unidirectional and Conjugately Matched Devices

For convenience, we will summarize the main results of the foregoing theory.

1) Each port of a general linear three-port network has in general two characteristic terminations which make the network unidirectional between the other two ports. They are designated cyclic or anticyclic according to the direction of the *nonzero* gain.

2) In a degenerate three-port network the two characteristic terminations of each port are equal.

3) Cyclic termination of one port of a three-port network and anticyclic termination of a second port produces critical stability.

4) In a *time-varying reactance* three-port network, the two characteristic terminations of a port have equal imaginary parts and equal and opposite real parts. Hence, only one is passive.

5) In a *time-varying reactance* three-port network, the passive characteristic terminations are either a) the cyclic ones at positive-frequency ports and the anticyclic ones at negative-frequency ports, or b) vice versa.

6) In a *time-varying reactance* three-port network, two ports terminated either both cyclically or both anticyclically are conjugately matched.

In applying the theory, we shall first decide which of the two characteristic terminations of a port is to be used, and then decide whether a positive or negative frequency must be specified for that port to make the required termination passive. The only restrictions on the time-varying circuit itself are that it must not be degenerate and that $\operatorname{Im} C_{12}C_{23}C_{31}$ must have the correct sign.

A. Time-Varying Reactance Three-Port Circulator

The circulator [2], [4] [Fig. 2(a)] is required to transmit energy from port 1 to 2, 2 to 3, and 3 to 1, and to have zero transmission in the reverse direction. Three cyclic characteristic terminations are therefore required, and for these to be passive the frequencies of the ports must be all positive or all negative (which amounts to the same thing). Defining the frequencies as positive, we have the matrix constraint for the clockwise circulator

$$\operatorname{Im} C_{12}C_{23}C_{31} > 0. \qquad (22)$$

The circulator will automatically be conjugately matched. The three frequencies may be different or, if a suitable circuit configuration is adopted, the same. In the latter case the device becomes exactly equivalent to a conventional ferrite circulator. The power gain from port n to port m is ω_m/ω_n. This follows directly from the Manley-Rowe relations, since no energy emerges at the third port.

B. Two-Port Amplifiers

Desirable properties of an amplifying device are: 1) conjugate matching at the input, 2) conjugate matching at the output, 3) unidirectionality, and 4) high gain, which in the context will be taken to mean arbitrarily high gain, i.e., the device may approach instability. Unfortunately, all four of these are not achievable simultaneously using a three-port time-varying reactance circuit.

a) Unidirectional conjugately matched upconverter. The time-varying reactance three-port circulator can also be used as a unidirectional conjugately matched upconverter by taking for example port 1 as the input and port 2 as the output. The forward power gain will be the same as for a simple upper-sideband converter, i.e., ω_2/ω_1.

b) Unidirectional high-gain converter with conjugately matched input [8] [*Fig. 2(b)*]. To obtain zero reverse gain we must terminate port 3 cyclically. To obtain a conjugate match at the input we must therefore also terminate port 1 cyclically. For passive terminations, frequencies ω_1 and ω_3 must thus have the same sign, which we will take as positive. The condition for high gain, since we have already terminated two ports cyclically, is that termination 2 must tend to its anticyclic value. For termination 2 to be passive, therefore, ω_2 must be negative. The output conductance will of course be negative. The matrix constraint is

$$\mathrm{Im}\; C_{12}C_{23}C_{31} > 0. \tag{23}$$

c) Unidirectional high-gain converter with conjugately matched output [*Fig.* 2(*c*)]. This is designed similarly to the circuit of Fig. 2(b). Port 3 is terminated cyclically for unidirectionality, port 2 also cyclically to give a conjugate match, and port 1 anticyclically for high gain. The matrix constraint is

$$\mathrm{Im}\; C_{12}C_{23}C_{31} > 0. \tag{24}$$

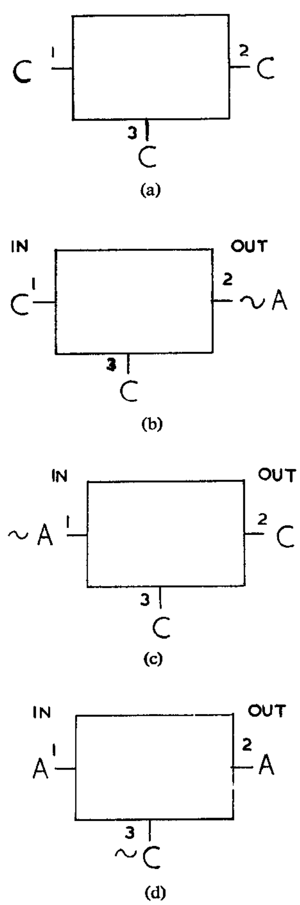

Fig. 2. Applications of the theory. (a) Circulator. (b) Unidirectional high-gain converter with conjugately matched input. (c) Unidirectional high-gain converter with conjugately matched output. (d) Conjugately matched high-gain converter. C denotes a cyclic characteristic termination, A an anticyclic characteristic termination.

d) Conjugately matched high-gain converter [*1*], [*8*], [*9*] [*Fig.* 2(*d*)]. To obtain a conjugate match at both input and output, we terminate ports 1 and 2 with their anticyclic characteristic terminations. Frequencies ω_1 and ω_3 must therefore both have the same sign, which we will take as positive. High gain can now be achieved by letting termination 3 tend to its cyclic value, and for this to be passive ω_3 must be negative. The gain can be controlled by varying Y_3 without destroying the conjugate matches at the other two ports. The reverse power gain, which is independent of Y_3, is the Manley-Rowe gain ω_1/ω_2, since there is no power flow from port 2 to port 3. The matrix constraint is

$$\mathrm{Im}\; C_{12}C_{23}C_{31} < 0. \tag{25}$$

It will be seen that applications b), c), and d) are essentially the same circuit with the ports numbered differently.

IV. Practical Example

As an illustration we will show how the theory may be applied to a circuit described by Adams [1], which is a conjugately matched high-gain upconverter [application in Section III-B-d]. The circuit (Fig. 3) consists of a time-varying capacitance $C(t)$ in series with three parallel resonant filters tuned to the input frequency ω_1, the output frequency ω_2, which is the upper sideband $\omega_1+\omega_p$, and the idler frequency ω_3, which is the lower sideband $\omega_1-\omega_p$. In the analysis the filters are assumed to be open circuit at their resonant frequencies and short circuit at all other frequencies, so that the voltage across the diode is made up only of the three components V_1, V_2, and V_3, having frequencies ω_1, ω_2, and ω_3, respectively. The admittance matrix of the network is obtained by the usual procedure [12], and can be shown to be

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} j\omega_1 C_0 & j\omega_1 C_1^* & j\omega_1 C_1 \\ j\omega_2 C_1 & j\omega_2 C_0 & j\omega_2 C_2 \\ j\omega_3 C_1^* & j\omega_3 C_2^* & j\omega_3 C_0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{26}$$

where C_0, C_1, and C_2 are components of $C(t)$ defined by the relation

$$C(t) = \cdots C_2^* e^{-2j\omega_p t} + C_1^* e^{-j\omega_p t} + C_0 + C_1 e^{j\omega_p t} + C_2 e^{2j\omega_p t} + \cdots. \tag{27}$$

The elements of this matrix can now be substituted in the general matrix of (4). The characteristic terminations are then obtained from (16) and (17) and their cyclic permutations, giving

$$Y_1(\text{anticyclic}) = -j\omega_1 C_0 + \frac{j\omega_1 (C_1^*)^2}{C_2^*} \tag{28}$$

$$Y_2(\text{anticyclic}) = -j\omega_2 C_0 + \frac{j\omega_2 C_1^* C_2}{C_1} \tag{29}$$

$$Y_3(\text{cyclic}) = -j\omega_3 C_0 + \frac{j\omega_3 C_1 C_2^*}{C_1^*}. \tag{30}$$

In practice the reactive parts of these admittances may be obtained by off-tuning the filters. The condition for the above terminations to be passive is, from (25),

$$\mathrm{Im}\; (C_1^*)^2 C_2 < 0. \tag{31}$$

Thus, both C_1 and C_2 must be nonzero.

The same circuit, with the same terminations, may also be used as a unidirectional high-gain converter with conjugately matched input by taking port 1 as the input, port 3 as the output, and port 2 as the idler port. Alternatively, a unidirectional high-gain converter with conjugately matched output results if we take port 3 as the input, port 2 as the output, and port 1 as the idler port. If inequality (31) were reversed, it would be possible to terminate ports 1 and 2 cyclically and port 3 anticyclically. The device would then act as a conjugately matched high-gain downconverter with input at port 2 and output at port 1.

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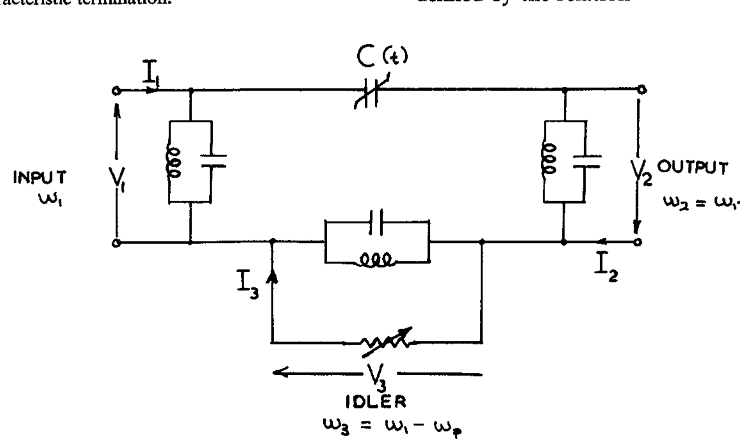

Fig. 3. Circuit of conjugately matched high-gain converter.

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